Differential Geometry III: Riemannian geometry

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19 Feb. 2025

1.1 Let $\mathcal{M}$ be a differentiable manifold of dimension m. Assume that $\mathcal{U}, \mathcal{U}'$ are two open subsets of $\mathcal{M}$ with $\mathcal{U} \cap \mathcal{U}' \neq \emptyset$, equipped with coordinate charts $\phi : \mathcal{U} \to \mathcal{V} \subset \mathbb{R}^m$ and $\phi' : \mathcal{U}' \to \mathcal{V}' \subset \mathbb{R}^m$. Let $(x^1, \dots, x^m)$ and $(y^1, \dots, y^m)$ be the corresponding coordinates on $\mathcal{U}$ and $\mathcal{U}'$, respectively; recall that each coordinate function $x^i : \mathcal{U} \to \mathbb{R}$ is defined so that

$$x^i = \bar{x}^i \circ \phi,$$

where $\bar{x}^i : \mathbb{R}^m \to \mathbb{R}$ is the projection on the i-th coordinate (equivalently, for any $q \in \mathcal{U}$, $x^i(q)$ is equal to the i-th component of the vector $\phi(q) \in \mathbb{R}^m$); similarly for y^i (with ϕ' in place of ϕ).

(a) Prove that the functions

$$\tilde{y}^i = y^i \circ \phi^{-1}, \quad i = 1, \dots, m$$

are differentiable functions on $\phi(\mathcal{U} \cap \mathcal{U}') \subset \mathbb{R}^m$ (*Hint: Use the assumption on the smoothness of transition functions on $\mathcal{M}$*). Show also that

$$\tilde{y}^i(x^1(p), \dots, x^m(p)) = y^i(p) \quad \text{for all } p \in \mathcal{U} \cap \mathcal{U}'$$

Remark. We usually refer to the function $\tilde{y}^i$ as the expression of the coordinate function y^i with respect to the $(x^1, \dots, x^m)$ coordinate system on $\mathcal{U} \cap \mathcal{U}'$.

(b) Show that the coordinate tangent vectors $\left\{\frac{\partial}{\partial x^i}\right\}_{i=1}^m$ and $\left\{\frac{\partial}{\partial y^i}\right\}_{i=1}^m$ satisfy at every point on $\mathcal{U} \cap \mathcal{U}''$

$$\frac{\partial}{\partial x^i} = \partial_i \tilde{y}^j \circ \phi \cdot \frac{\partial}{\partial y^j}.$$

Find a similar relation between the coordinate covectors $\{dx^i\}_{i=1}^m$ and $\{dy^i\}_{i=1}^m$.

1.2 Construct a smooth atlas (not necessarily maximal) on the unit sphere

$$\mathbb{S}^n = \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}.$$

1.3 Let $(\mathcal{M}, g)$ be a Riemannian manifold and let $(x^1, \dots, x^n)$ and $(y^1, \dots, y^n)$ be two systems of coordinates around a point $p \in \mathcal{M}$. Let g_{ij} be the components of the metric g in the $(x^1, \dots, x^n)$ coordinates, while $\tilde{g}_{ij}$ are the components of g with respect to $(y^1, \dots, y^n)$. Show that

$$\tilde{g}_{ij} = \frac{\partial x^a}{\partial y^i} \frac{\partial x^b}{\partial y^j} g_{ab},$$

where, in the above expressions, the coordinate functions x^γ are considered as functions of $(y^1, \dots, y^n)$ (see Ex. 1.1). Express the Euclidean metric on $\mathbb{R}^2 \setminus \{0\}$ in polar coordinates.

1.4 Let $\mathcal{M}$ be a differentiable manifold and $F : \mathcal{M} \to \mathbb{R}^N$ be an immersion. The metric g *induced* on $\mathcal{M}$ by the Euclidean metric on $\mathbb{R}^N$ is defined by the relation

$$g(X, Y) \doteq \langle dF(X), dF(Y) \rangle_{\mathbb{R}^N} \quad \text{for all tangent vectors} X, Y \text{ on } \mathcal{M}.$$

(a) Show that, in any local coordinate system $(x^1, \dots, x^n)$ on $\mathcal{M}$, the components of g are given by

$$g_{ij} = \delta_{ab} \frac{\partial F^a}{\partial x^i} \frac{\partial F^b}{\partial x^j}.$$

(b) **(Surface of revolution)** Let $\gamma : (0,1) \to \mathbb{R}^2$ be a smooth curve parametrized with unit speed (i.e. $\langle \frac{d\gamma}{du}(u), \frac{d\gamma}{du}(u) \rangle = 1$). Let $\gamma(u) = \big(X(u), Y(u)\big)$ be the representation of γ in the standard Cartesian coordinates on $\mathbb{R}^2$ and assume that $X(u) > 0$ for all $u \in (0,1)$. Consider the surface of revolution $\mathcal{S} \subset \mathbb{R}^3$ obtained by rotating the curve γ around the y-axis; this surface is parametrized by $(u, \theta) \in (0,1) \times [0, 2\pi)$ via the map

$$\Psi(u, \theta) = \big(X(u)\cos\theta,\, Y(u),\, X(u)\sin\theta\big).$$

Express the induced metric on $\mathcal{S}$ from the Euclidean metric on $\mathbb{R}^3$ in the (u, θ) coordinates.

1.5 Let $(\mathcal{M}, g)$ be a Riemannian manifold and $x^1, \dots, x^n$ a system of local coordinates on an open subset $U \subset \mathcal{M}$ associated to a coordinate chart $\phi : U \to \mathbb{R}^n$. Show that the volume

$$\mathrm{Vol}(U) = \int_{\phi(U)} \sqrt{\det(g_{ij})}\, \mathrm{d}x^1 \dots \mathrm{d}x^n$$

is independent of the choice of coordinates.